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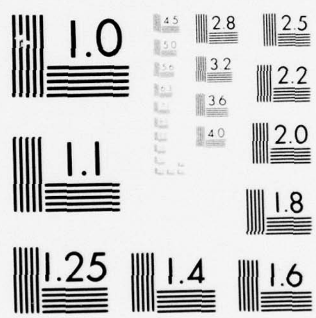
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(6) SOME APPROXIMATIONS IN MULTI-ITEM, MULTI-ECHELON INVENTORY SYSTEMS FOR RECOVERABLE ITEMS

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SUMMARY

The optimization problem as formulated in the METRIC model ~~[3]~~ takes the form of minimizing the expected number of total system backorders in any two-echelon inventory system subject to a budget constraint. To solve this problem, one needs to find the optimal Lagrangian multiplier* associated with the given budget constraint.

For any large scale inventory system, this task is computationally not trivial. Fox and Landi proposed one method which was a significant improvement over the original METRIC algorithm. In this report we first develop a method for estimating the value of the optimal Lagrangian multiplier used in the Fox-Landi algorithm is developed, ~~present~~ alternative ways for determining stock levels are presented, and ~~compare~~ these proposed approaches with the Fox-Landi algorithm are compared using two hypothetical inventory systems--one involving 3 bases and 75 items; the other has 5 bases and 125 items. The comparison shows that the computational time can be reduced by nearly 50 percent.

Another factor that contributes to the higher requirement for computational time in obtaining the solution to two-echelon inventory systems is that it has to optimally allocate stock to the depot as well as to bases for a given total system stock level. This essentially requires the evaluation of every possible combination of depot and base stock levels--a time-consuming process for many practical inventory problems with a sizable system stock level. This report also suggests a simple approximation method for estimating the optimal depot stock level. When this method was applied to the same two hypothetical inventory systems indicated above, it was found that the estimate of optimal depot stock is quite close to the optimal value in all cases. Furthermore, the increase in expected system backorders using the estimated depot stock levels rather than the optimal levels is generally small.

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*The economic interpretation of a Lagrangian multiplier is a reduction in backorder or shortage that can come about because of an increase in the investment in inventory.

Further research will be required to estimate more precisely the reduction in computational time if these approximation methods are incorporated in the requirements computation system for recoverable spares (D041) by the Air Force. For instance, we would have to apply the proposed methods to stratified samples of recoverable items from the D041 system and extrapolate the results on an Air Force-wide basis. Nevertheless, even on qualitative grounds, the proposed methods are so simple and reasonably accurate, our conclusion is that its implementation will be beneficial.

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I. INTRODUCTION

Almost a decade ago Sherbrooke formulated the well-known METRIC model for determining optimal stock levels for recoverable items--items subject to repair when they fail--in a two-echelon setting [3]. Briefly, the two-echelon system he studied consists of several locations, called bases, at which primary demands occur; these bases are in turn resupplied as necessary by a central repair and inventory stocking point called a depot. When a failure occurs at a base, a demand is placed on base supply for a corresponding replacement part. The failed part is then either repaired at that base or is sent to the depot for repair depending on the nature of the failure. Resupply of base supply comes from the base maintenance organization if repair is accomplished at the base; otherwise, resupply comes from the depot. In either case, the organization resupplying the base supply activity does so by exchanging a serviceable part for the failed part. Thus the inventory policy for placing orders on the base's maintenance organization or the depot is an (s - 1,s) policy.

Sherbrooke presented a model for determining both depot and base stock levels for all items for this system. In particular, the problem he formulated was to minimize the average total number of base backorders existing at an arbitrary point in time subject to a constraint on system investment, that is,

$$\min \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{x>s_{ij}} (x - s_{ij})p(x|\lambda_{ij}T_{ij}(s_{i0}))$$

$$\text{subject to} \quad \sum_{j=0}^{m} \sum_{i=1}^{n} c_i s_{ij} \leq C , \qquad \text{(P1)}$$

where n = the number of items,

m = the number of bases,

s_{ij} = the stock level at base j for item i,

s_{i0} = the depot stock level for item i,

λ_{ij} = the expected daily demand rate for item i at base j,

c_i = the unit cost for item i,

C = the budget constraint,

$T_{ij}(s_{i0})$ = the average resupply time for base j for item i given the depot stock level for item i is s_{i0}, and

$p(x|y)$ = the probability that x units are in the resupply system given that the expected number of units in the resupply system is y.

Furthermore, Sherbrooke shows that $T_{ij}(s_{i0})$ can be expressed as

$$T_{ij}(s_{i0}) = r_{ij}A_{ij} + (1 - r_{ij})(B_{ij} + \delta(s_{i0}) \cdot D_i) ,$$

where A_{ij} = the average base repair time for item i at base j,

r_{ij} = the proportion of demands requiring base repair for item i at base j,

B_{ij} = the average order-and-ship time at base j for item i,

D_i = the average depot repair cycle time for item i,

$\delta(s_{i0}) \cdot D_i = (1/\lambda_i) \sum_{x>s_{i0}} (x - s_{i0})p(x|\lambda_i D_i)$, the expected delay per depot demand for item i, and

$\lambda_i = \sum_{j=1}^{m} (1 - r_{ij})\lambda_{ij}$, the expected daily depot demand for item i.

In the remainder of the report, i will refer to an item and j will refer to a base (j = 0 represents the depot). Thus i and j will always be elements of the sets {1, ..., n} and {0, ..., m}, respectively. Additionally, an integer k appearing in the text to the right of the statement of a problem or equation will designate for future reference that problem or equation. For complete description of problem background and formulation, see Ref. 3.

Subsequently Fox and Landi suggested a Lagrangian approach for solving problem P1 [2]. One obstacle to the implementation of METRIC using the Fox-Landi algorithm is the requirement of estimating an appropriate value for the Lagrangian multiplier. Another important and related problem is the lengthy computer run time required to obtain an optimal solution to problem P1 when using their algorithm. A large portion of this computational effort is related to searching for the optimal depot stock level. This search is particularly time-consuming for items having a high average number of units in the depot repair cycle since the amount of computation required by their algorithm is roughly proportional to the number of depot stock levels explicitly examined.

In this report we first develop an approach for obtaining an estimate of the optimal Lagrange multiplier value required in the Fox-Landi algorithm, present two new methods for determining stock levels, and compare these methods with the Fox-Landi method and other techniques. The proposed approach eliminates the particularly time-consuming portion of the Fox-Landi algorithm devoted to searching for the best Lagrange multiplier value and significantly reduces computation time for determining stock levels without degrading the quality of the solution.

We then present a method for estimating the optimal depot stock level. Limited computational experience indicates that this method is easy to implement, provides a very good estimate of the optimal depot stock level, and is particularly useful for items having a high average number of units in the depot repair cycle. For these items it is possible to reduce computation time required by the Fox-Landi algorithm by as much as 90 percent.

II. THE APPROXIMATION PROBLEM

In this section we first construct a problem that is a continuous approximation to problem P1. We then state and prove two theorems that are the basis for an algorithm that can be used to solve this approximating problem.

Recall that the total average base backorders existing at any point in time for item i can be expressed as

$$\sum_{j=1}^{m} \sum_{x>s_{ij}} (x - s_{ij}) p(x|\lambda_{ij} T_{ij}(s_{i0})) .$$

Two useful probability distributions for describing the demand process are the Poisson and negative binomial distributions. As shown in Ref. 1, this implies that if demand has a Poisson or negative binomial distribution, then for a given value of $\lambda_{ij}T_{ij}(s_{i0})$, $p(x|\lambda_{ij}T_{ij}(s_{i0}))$, the probability distribution representing the number of units in resupply of item i at base j at any point in time, is a Poisson or negative binomial distribution, respectively.

Experimental data gathered during the conduct of this study indicate that when $p(x|\lambda_{ij}T_{ij}(s_{i0}))$ is either a Poisson or negative binomial distribution, the above total expected backorder expression can be closely approximated by an exponential function. That an exponential function accurately approximates this expression should not be entirely unexpected. First, for budgets of practical interest, the item stock levels, s_{ij}, are normally much larger than the average demand during the resupply time. In fact, the probability of running out of stock during the resupply time is often much less than .15 in real applications. Thus the only probabilities entering the backorder calculation are the tail probabilities of the distribution. In the tails, both the Poisson and negative binomial distributions behave almost like the geometric distribution; that is, each succeeding probability is roughly a constant proportion of its predecessor. Consequently, when s_{ij} is large relative to $\lambda_{ij}T_{ij}(s_{i0})$, the expected number of backorders existing at any time at location j for item i is approximately a geometric function of s_{ij}. Therefore, an exponential function is a useful continuous approximation to this relationship between expected backorders at a location and the item's stock level at that location.

Furthermore, total expected base backorders exhibit this same behavior. If demand has either a Poisson or negative binomial distribution (or, for that matter, any compound Poisson distribution), then the total number of units of an item in resupply across all bases has a Poisson or negative binomial distribution, respectively, given we assume independence of demand and common variance-to-mean ratio among bases. Since in most practical situations total system stock substantially exceeds the total expected number of units in resupply, the tail of the distribution describing the total number of units in resupply is the only portion of the distribution of importance. As an approximation, this distribution can be used to determine the nature of the relationship between total expected base backorders and total system stock. For the reasons discussed previously, an exponential function should also adequately represent this relationship.

Thus we will approximate

$$\sum_{j=1}^{m} \sum_{x>s_{ij}} (x - s_{ij})p(x|\lambda_{ij}T_{ij}(s_{i0}))$$

with the exponential function

$$B_i(N_i) \equiv a_i e^{-b_i N_i}$$

In this approximation, N_i represents total system stock. In practice, the parameters $a_i > 0$ and $b_i > 0$ are estimated using regression analysis. The data used in the regression analysis are the backorder data obtained from the solution to the problem

$$\min \sum_{j=1}^{m} \sum_{x>s_{ij}} (x - s_{ij})p(x|\lambda_{ij}T_{ij}(s_{i0}))$$

$$\text{subject to} \quad \sum_{j=0}^{m} s_{ij} = N_i \ , \quad \text{and}$$

$$s_{ij} = 0, 1, \ldots, N_i \ ,$$

for several appropriate values of N_i.

We now formulate a continuous approximation to problem P1 in which the exponential representation of total system backorders for an item is used. In this approximation problem, the decision variables are the total system stock, N_i, rather than the stock levels for each location, s_{ij}. As we shall see, the main reason for studying this approximation problem is that it is a vehicle for obtaining an estimate of the optimal Lagrangian multiplier value used in the Fox-Landi algorithm. The approximation problem is formulated as problem P2:

$$\min \sum_{i=1}^{n} B_i(N_i)$$

$$\text{subject to} \quad \sum_{i=1}^{n} c_i N_i \leq C \ , \tag{P2}$$

$$\text{where} \quad N_i \geq 0 \ .$$

Note that N_i is a continuous variable in this approximation. The optimality conditions (Kuhn-Tucker conditions) for this problem are as follows:

Find $\theta_1 \geq 0$ such that

$$\text{(a)} \quad \frac{dB_i}{dN_i} + \theta_1 c_i \geq 0 \ ,$$

(b) $\sum_{i=1}^{n} c_i N_i \leq C$,

$N_i \geq 0$,

(c) $\theta_1 \left(\sum_{i=1}^{n} c_i N_i - C \right) = 0$,

(d) $N_i \left(\frac{dB_i}{dN_i} + \theta_1 c_i \right) = 0$.

A relaxed version of problem P2 in which the non-negativity constraint on the item stock level is removed is problem P3:

$$\min \sum_{i=1}^{n} B_i(N_i)$$

$$\text{subject to} \quad \sum_{i=1}^{n} c_i N_i \leq C . \qquad \text{(P3)}$$

The optimality conditions for this problem are:

Find $\theta_2 \geq 0$ such that

(a) $\frac{dB_i}{dN_i} + \theta_2 c_i = 0$,

(b) $\sum_{i=1}^{n} c_i N_i \leq C$,

(c) $\theta_2 \left(\sum_{i=1}^{n} c_i N_i - C \right) = 0$,

(d) $N_i \left(\frac{dB_i}{dN_i} + \theta_2 c_i \right) = 0$.

We now explore the relationship between problems P2 and P3 in detail.

Suppose we obtained a solution to problem P3.* Let N_i^1 represent the optimal solution to problem P2, and N_i^2 represent the optimal solution to problem P3. If $N_i^2 \geqq 0$ for all i, then $N_i^1 = N_i^2$ and the objective function values are equal.

Suppose, however, that $N_i^2 < 0$ for at least one value of i. Let

$$\overline{N}_i = \max(0,\ N_i^2)$$

and

$$\overline{C} \equiv \sum_{i=1}^{n} c_i \overline{N}_i \ .$$

Since $\overline{N}_i \geqq N_i^2$ for all i and $\overline{N}_i > N_i^2$ for at least one i, $\overline{C} > C$.

Suppose problem P2 is modified slightly so that the right-hand side value C is replaced by $\overline{C}$. This modified problem is problem P4:

$$\min \sum_{i=1}^{n} B_i(N_i)$$

$$\text{subject to} \quad \sum_{i=1}^{n} c_i N_i \leqq \overline{C} \ , \qquad \text{(P4)}$$

$$\text{where} \quad N_i \geqq 0 \ .$$

The optimality conditions for this problem are the same as those given for problem P2 after substituting $\overline{C}$ for C. Also, let $\overline{\theta}$ represent the optimal value of the Lagrangian multiplier for problem P4.

In solving problem P3, we will obtain a value for θ_2. We now show that $\overline{\theta} = \theta_2$, and that $\overline{N}_i = \max(0,\ N_i^2)$ is an optimal solution to

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*Section III develops the method for determining the solution to problem P3.

problem P4 by demonstrating that these values satisfy the Kuhn-Tucker conditions corresponding to problem P4.

By construction,

$$\sum_{i=1}^{n} c_i \overline{N}_i = \overline{C}\ , \quad \overline{N}_i \geqq 0\ , \quad \text{and} \quad \overline{\theta}\left(\sum_{i=1}^{n} c_i \overline{N}_i - \overline{C}\right) = 0\ .$$

If $\overline{\theta} = \theta_2$, $\overline{\theta} \geqq 0$ since $\theta_2 \geqq 0$. Suppose $\overline{N}_i = N_i^2$, that is, $N_i^2 \geqq 0$. Then

$$\left.\frac{dB_i}{dN_i}\right|_{N_i=\overline{N}_i} = \left.\frac{dB_i}{dN_i}\right|_{N_i=N_i^2}$$

and

$$0 = \left.\frac{dB_i}{dN_i}\right|_{N_i=N_i^2} + \theta_2 c_i = \left.\frac{dB_i}{dN_i}\right|_{N=\overline{N}_i} + \overline{\theta} c_i\ .$$

By assumption there exists at least one value of i for which $\overline{N}_i > N_i^2$; that is, $\overline{N}_i = 0$ while $N_i^2 < 0$. Since

$$\left.\frac{dB_i}{dN_i}\right|_{N_i=0} > \left.\frac{dB_i}{dN_i}\right|_{N_i<0}\ ,$$

due to the exponential form of $B_i(N_i)$, and

$$\left.\frac{dB_i}{dN_i}\right|_{N_i=N_i^2} + \theta_2 c_i = 0\ ,$$

we know that

$$\left.\frac{dB_i}{dN_i}\right|_{N_i=0} + \bar{\theta}c_i > 0 \ .$$

Consequently, the optimal solution to problem P4 is $N_i = \bar{N}_i = \max\{0, N_i^2\}$. Furthermore, the optimality conditions are satisfied when $\bar{\theta}$ is equal to θ_2.

Theorem 1. $\theta_1 \geqq \theta_2$.

Proof: The optimal objective function value for problem P2 is a convex, differentiable, strictly decreasing function of the available budget, C. Since the slope of this function is equal to the negative of the Lagrangian multiplier value, $\theta_1 \geqq \bar{\theta}$ since $C \leqq \bar{C}$. But $\theta_2 = \bar{\theta}$, so $\theta_1 \geqq \theta_2$.

Corollary. $\theta_1 > \theta_2$ *when* $\bar{C} > C$.

Next we compare N_i^1 with $\bar{N}_i$. If $C = \bar{C}$, then $N_i^1 = \bar{N}_i$ for all i. Now let us suppose $\bar{C} > C$ so that $\theta_1 > \theta_2 = \bar{\theta}$. Let us examine the two cases $\bar{N}_i > 0$ and $\bar{N}_i = 0$ separately.

First, assume $\bar{N}_i > 0$. Then

$$\left.\frac{dB_i}{dN_i}\right|_{N_i=\bar{N}_i} + \bar{\theta}c_i = 0 \ .$$

Furthermore, if $N_i^1 > 0$, then

$$\left.\frac{dB_i}{dN_i}\right|_{N_i=N_i^1} + \theta_1 c_i = 0 \ .$$

Since

$$\theta_1 c_i > \bar{\theta}c_i = -\left.\frac{dB}{dN_i}\right|_{N_i=\bar{N}_i} ,$$

$$\left.\frac{dB_i}{dN_i}\right|_{N_i=\overline{N}_i} > \left.\frac{dB_i}{dN_i}\right|_{N_i=N_i^1} ,$$

and $N_i^1 < \overline{N}_i$. If $N_i^1 = 0$, then $\overline{N}_i > N_i^1$.

Next assume $\overline{N}_i = 0$. Since

$$\left.\frac{dB_i}{dN_i}\right|_{N_i=0} + \theta_1 c_i > \left.\frac{dB_i}{dN_i}\right|_{N_i=0} + \overline{\theta} c_i \geqq 0 ,$$

it follows that $N_i^1 = 0$ by complementary slackness. Thus we have proven the following theorem.

Theorem 2. $\overline{N}_i \geqq N_i^1$; *additionally*, $\overline{N}_i > N_i^1$ *whenever* $\overline{N}_i > 0$.

In this section we established several important relationships among problems P2, P3, and P4. In Sec. III we develop a simple algorithm for solving problem P2 based on these relationships and show how to find the solution to problem P3. As we have just demonstrated, once we have the solution to problem P3, we also have the solution to problem P4. From Theorem 2, we then have an upper bound on the values of the N_i^1. In particular, if $\overline{N}_i = 0$, then $N_i^1 = 0$. Combining this observation with the implications of Theorem 1 and its corollary provides the basis for the proposed algorithm for solving problem P2.

III. COMPUTING OPTIMAL SOLUTIONS FOR PROBLEMS P2 AND P3

We begin this section by developing a method for determining the optimal solution to problem P3. Observe that the optimal solution must satisfy the following two conditions:

$$\frac{dB_i}{dN_i} + \theta_2 c_i = 0$$

and

$$\sum_{i=1}^{n} c_i N_i = C \ .$$

The second condition must hold since each $B_i(N_i)$ is a strictly decreasing function of N_i.

Since

$$B_i(N_i) = a_i e^{-b_i N_i} \ ,$$

where $a_i, b_i > 0$, the first condition states that

$$\theta_2 = \frac{a_i b_i e^{-b_i N_i}}{c_i} > 0 \ ,$$

or

$$\hat{\theta} \triangleq \ln \theta_2 = \ln\left(\frac{a_i b_i}{c_i}\right) - b_i N_i \ .$$

Letting

$$d_i = \ln\left(\frac{a_i b_i}{c_i}\right) ,$$

we see that

$$N_i = \frac{d_i - \hat{\theta}}{b_i} .$$

From the second condition we know that

$$\sum_{i=1}^{n} c_i \left(\frac{d_i - \hat{\theta}}{b_i} \right) = C .$$

Thus

$$\hat{\theta} = \frac{\sum_{i=1}^{n} (c_i d_i / b_i) - C}{\sum_{i=1}^{n} (c_i / b_i)} .$$

Letting

$$\alpha = \sum_{i=1}^{n} \frac{c_i d_i}{b_i} \quad \text{and} \quad \beta = \sum_{i=1}^{n} \frac{c_i}{b_i} ,$$

we can express $\hat{\theta}$ as

$$\hat{\theta} = \frac{\alpha - C}{\beta} .$$

Thus

$$\theta_2 = e^{(\alpha - C)/\beta} \tag{E1}$$

and

$$N_i = \frac{d_i - \frac{\alpha - C}{\beta}}{b_i} = \frac{g_i + C}{f_i} , \tag{E2}$$

where $g_i = \beta d_i - \alpha$ and $f_i = \beta b_i$. Consequently, N_i is a *linear function of* C. If the budget is incremented by an amount ΔC, then N_i', the new value of the stock level for item i, satisfies

$$N_i' = N_i + \frac{\Delta C}{f_i} .$$

The optimal solution to problem P2 has been found if each of the N_i found using Eq. (E2) is non-negative. If there exists an i for which $N_i < 0$, then we may employ the following algorithm to find the optimal solution to problem P2. Let $I = \{1, \ldots, n\}$ and N_i^1 represent the optimal solution to problem P2.

Step 0. Solve Problem P3 as described above, thereby obtaining an initial value for N_i, $i \in I$.

Step 1. Set $N_i^1 = 0$ for all $N_i < 0$ during the last iteration and delete the corresponding i from I. Recompute α and β, where

$$\alpha = \sum_{i \in I} \left\{ \frac{c_i d_i}{b_i} \right\}$$

and

$$\beta = \sum_{i \in I} \left\{ \frac{c_i}{b_i} \right\} .$$

Step 2. Using Eq. (E2), obtain new estimates of N_i for each $i \in I$. If $N_i \geq 0$ for all $i \in I$, then the optimal solution has been found, and $N_i^1 = N_i$ for all $i \in I$ and $N_i^1 = 0$ for all $i = 1, \ldots, n$ for which $i \notin I$. If there exists some i for which $N_i < 0$, return to step 1.

It is clear that our solution satisfies all the optimality conditions for problem P2 except possibly condition (a) for $i \notin I$. However, at an earlier iteration (when i was deleted from I) we had

$$\left.\frac{dB_i}{dN_i}\right|_{N_i=\tilde{N}_i} + \tilde{\theta}_2 c_i = 0 \ ,$$

where $\tilde{\theta}_2$ and $\tilde{N}_i(<0)$ are the earlier values of θ_2 and $\tilde{N}_i$, respectively. Since dB_i/dN_i is clearly increasing in N_i, and θ_2 increases at each iteration (Theorem 1 and its corollary), condition (a) must hold. Convergence is guaranteed since n is finite.

IV. A COMPARISON OF ALTERNATIVE SOLUTION PROCEDURES FOR SOLVING PROBLEM P1

In this section we review three algorithms for solving problem P1 and compare them to two algorithms designed to obtain a solution for problem P1 based on the solution to the approximating problem, problem P2.

THE SHERBROOKE PROCEDURE

The first algorithm, a procedure originally proposed by Sherbrooke [3], is a marginal analysis algorithm consisting of two phases. In the first phase, each item is examined independently. The optimization problem solved for item i in the first phase has the form:

$$\min \sum_{j=1}^{m} \sum_{x>s_{ij}} (x - s_{ij})p(x|\lambda_{ij}T_{ij}(s_{i0}))$$

$$\text{subject to} \quad \sum_{j=0}^{m} s_{ij} = N_i , \tag{P5}$$

$$\text{where} \quad s_{ij} = 0, 1, \ldots,$$

and N_i is the total system stock available for distribution among the depot and bases. Let $Z_i(N_i)$ represent the optimal objective function value for problem P5 given N_i units are available for distribution. Problem P5 is solved by obtaining the solution to the $N_i + 1$ problems

$$\overline{Z}_i(N_i, s_{i0}) = \min \sum_{j=1}^{m} \sum_{x>s_{ij}} (x - s_{ij})p(x|\lambda_{ij}T_{ij}(s_{i0}))$$

$$\text{subject to} \quad \sum_{j=1}^{m} s_{ij} = N_i - s_{i0} , \tag{P6}$$

$$\text{where} \quad s_{ij} = 0, 1, \ldots,$$

and s_{i0} is fixed for $s_{i0} = 0, 1, \ldots, N_i$. Problem P6 can be solved via marginal analysis. Then

$$Z_i(N_i) = \min_{s_{i0}} \bar{Z}_i(N_i, s_{i0}) ,$$

$$\text{where} \quad s_{i0} = 0, \ldots, N_i .$$

The second-phase problem is

$$\min \sum_{i=1}^{n} Z_i(N_i)$$

$$\text{subject to} \quad \sum_{i=1}^{n} c_i N_i \leq C ,$$

$$\text{where} \quad N_i = 0, 1, \ldots .$$

Sherbrooke [3] suggests that a marginal analysis algorithm be used to find a solution to this knapsack problem. Clearly other procedures could be employed to obtain an optimal solution. In any case, this approach requires a substantial amount of storage to save all the $Z_i(N_i)$ values. For moderately sized problems--several thousand items--a storage requirement of 10^6 or more words may be needed to save these values. Furthermore, the computation time required to obtain these $Z_i(N_i)$ values for such problems is very large.

THE FOX AND LANDI PROCEDURE

Subsequently Fox and Landi [2] proposed a Lagrangian algorithm for solving problem P1. In particular, they formulated the relaxed version of problem P1 as problem P7:

$$\min \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{x>s_{ij}} (x - s_{ij}) p(x|\lambda_{ij} T_{ij}(s_{i0})) + \theta \sum_{j=0}^{m} \sum_{i=1}^{n} c_i s_{ij} , \tag{P7}$$

where $s_{ij} = 0, 1, \ldots ,$

and θ is the Lagrangian multiplier. Since problem P7 is separable by item, its optimal solution can be found by solving the n individual item problems

$$\min \sum_{j=1}^{m} \sum_{x>s_{ij}} (x - s_{ij}) p(x|\lambda_{ij} T_{ij}(s_{i0})) + \theta \sum_{j=0}^{m} c_i s_{ij}$$

subject to $s_{ij} = 0, 1, \ldots .$

This problem, like problem P6 in Sherbrooke's two-phase method, is solved using a partitioning procedure, that is, it is reformulated as

$$\min_{s_{i0}=0,1,\ldots} \left\{ \theta c_i s_{i0} + \sum_{j=1}^{m} \min_{s_{ij}=0,1,\ldots} \left\{ \sum_{x>s_{ij}} (x - s_{ij}) p(x|\lambda_{ij} T_{ij}(s_{i0})) + \theta c_i s_{ij} : s_{i0} \text{ fixed} \right\} \right\} , \tag{P8}$$

or equivalently as

$$\min Z(s_{i0};\, \theta) \tag{P9}$$

where $s_{i0} = 0, 1, \ldots ,$

and

$$Z(s_{i0};\theta) = \theta c_i s_{i0} + \sum_{j=1}^{m} \min_{s_{ij}} \left\{ \sum_{x>s_{ij}} (x - s_{ij}) p(x|\lambda_{ij}T_{ij}(s_{i0})) \right.$$

$$\left. + \theta c_i s_{ij} : s_{ij} = 0, 1, \ldots ; s_{i0} \text{ fixed} \right\} .$$

To determine $Z(s_{i0};\theta)$, solve the m base problems

$$\min_{s_{ij}} \sum_{x>s_{ij}} (x - s_{ij}) p(x|\lambda_{ij}T_{ij}(s_{i0})) + \theta c_i s_{ij} .$$

The optimal s_{ij} is the smallest non-negative integer for which

$$\sum_{x>s_{ij}} p(x|\lambda_{ij}T_{ij}(s_{i0})) \leqq \theta c_i .$$

Problem P8 is solved for each item for a given value of θ. This yields a total investment cost corresponding to θ. In the Fox-Landi approach, the "optimal" value of θ is selected from a grid of M equally spaced values,

$$\theta_0 > \theta_1 > \ldots > \theta_M > 0 .$$

The optimal value of θ is the θ_K, $K \in \{0, \ldots, M\}$, whose corresponding total investment cost is closest to C.

Fox and Landi suggest that their method is a single-pass method; that is, only one pass through the item data base is necessary to obtain the optimal solution. The storage requirement to effect this one-pass approach is potentially enormous. For a moderately sized problem having 3000 items, 20 bases, and M = 63, almost 4 million item stock levels must be saved, plus possibly millions of additional item data elements reflecting fill rates, probability of no stockout at an arbitrary point in time, expected base backorders, etc. Furthermore, because there may be no simple method for estimating suitable bounds on the values of the multipliers, much larger values of M may be required to ensure adequate approximation of the budget.

It has been the author's experience that Air Force personnel have difficulty estimating a reasonable range for θ for large problems. This is not surprising because the data used in the model frequently change in real situations, thereby causing the optimal value of the multiplier to change. Furthermore, changing the multiplier's magnitude by 10^{-6} or less often causes the corresponding total cost to change by many millions of dollars. Consequently, 2^{10} values of θ have been used in some Air Force applications to make the system "foolproof." In these cases 60 million or more item stock levels would be needed to be explicitly stored--plus a considerable amount of other item and base data--to make the Fox-Landi algorithm truly a one-pass method.

On the other hand, if their method is altered so that the item data are passed through a second time, it is possible to eliminate virtually all the requirement for secondary storage. In the first pass, only the running total cost corresponding to each θ_K, $K \in \{0, \ldots, M\}$, is saved. At the end of this phase the "optimal" multiplier value, θ^*, is established. The second phase of the algorithm requires a second pass through the data base. In the second pass, the optimal stock levels for each location are found for all items by resolving problem P8 with $\theta = \theta^*$.

In some applications the Fox-Landi one-pass method is clearly infeasible; that is, there may not be enough peripheral storage capacity to save all the data. If storage capacity is available, there is a tradeoff between the time and cost required to store and access the data in the secondary memory using the one-pass method, and the time and cost to recompute the stock levels using the second method. For realistic Air Force problems, the two-pass method appears to be the only feasible approach given current hardware constraints if M is large enough to guarantee that a solution can be found that closely approximates the target budget.

THE BISECTION METHOD

A third way to solve problem P1 is a slight modification for the Fox-Landi algorithm called the bisection method, which employs a bisection search to find the optimal value for θ. This procedure requires initial upper and lower bounds on the optimal value of θ. Call these θ_U and θ_L, respectively. The bisection method is as follows:

1. Set $\bar{\theta} = (\theta_U + \theta_L)/2$.
2. Solve problem P8 with $\theta = \bar{\theta}$ for each item.
3. If the total cost of the solution obtained in step 2 exceeds C, then replace θ_L with $\bar{\theta}$; otherwise, replace θ_U with $\bar{\theta}$.
4. If a stopping criterion has not been met (such as a fixed number of iterations or an error tolerance), return to step 1; otherwise, stop.

The major drawback to the bisection approach is that a separate pass through the item data base is required at each iteration of the algorithm. This algorithm performs very well in terms of convergence, and we have found that it almost always produces solutions that are within 1/2 percent of the target budget using 10 bisections.

COMPARISON OF METHODS

The closeness of the solutions to the target budget generated by either the Fox-Landi method or the bisection algorithm depends on how broad a range of multiplier values must be searched for a fixed value of M or a fixed number of bisections. It should be pointed out that both of these methods only yield an approximation to the optimal multiplier value (assuming one exists).

Of the methods discussed thus far, it has been the experience of the author, as well as of Fox and Landi [2], that the latter two algorithms dominate Sherbrooke's algorithm in run times by an order of magnitude or more on real problems given reasonable estimates of upper and lower bounds for the Lagrangian multiplier. Thus in the comparisons we will report, only these two Lagrangian methods will be discussed.

APPROXIMATION METHODS

Earlier we described an approximation method for estimating the optimal values of θ and each N_i. Several options are open for implementing this approximation method. One way to implement it is to use a two-phase approach. Call this approach the First Approximation Method. The values of a_i and b_i are computed in the first phase of this method and the optimal value of θ is estimated using Eq. (E1). In the second phase, we solve problem P8 for each item, using the estimate of the optimal θ. This approach has two major advantages over the Fox-Landi method:

1. The estimate of the optimal multiplier can be obtained without prespecifying a range of values, and computation time to obtain the estimate does not depend on the uncertainty of the multiplier value.
2. The computation time to find an estimate of the optimal multiplier is much smaller.

If the two-pass version of the Fox-Landi algorithm is used, the second phase of that method and the approximation method are the same. The one-pass version of the Fox-Landi algorithm requires considerably more storage, and also requires more computer time to determine the optimal stock levels than this approximation method requires.

This approximation approach also has advantages over the bisection method:

1. Only two passes through the data base are required as opposed to seven or more required for the bisection method in practice.
2. No stock levels need to be saved; in the bisection method it is necessary to save all stock levels and other data for three multiplier values.

Another algorithm can be employed that directly uses the results of the approximation problem, that is, problem P2. Call this approach the Second Approximation Method. This algorithm is of interest in situations in which we only want to compute total system stock for each item and are not particularly interested in computing the optimal distribution of the assets. Determining the optimal allocation of a budget among items is of primary importance when purchasing inventory or making budgetary projections for spares for different systems. In these cases, distribution decisions are usually not that critical.

This Second Approximation algorithm also consists of two phases. In the first phase we estimate the values of the a_i and b_i parameters, and in the second phase we determine total system stock for each item using the algorithm described in Sec. III and rounding N_i to the nearest integer. The algorithm requires one pass through the item data base and one pass through an item file consisting of a_i, b_i, and c_i. The major advantage of this approach is that it eliminates the stock allocation phase of both the Fox-Landi method and the First Approximation algorithm.

V. A COMPUTATIONAL COMPARISON OF VARIOUS ALGORITHMS

The Fox-Landi algorithm, bisection algorithm, and the two approximation methods have been coded and tested on several sample sets of data for the Air Force's new F-15 fighter. Since all of the tests yielded the same general results, we will discuss only two of them in detail. The first test consisted of a 75-item sample and had 3 operating bases. The flying programs were very different at each base. In the second test, 125 items were included in the sample with demands occurring at 5 bases. In the second test, only the Fox-Landi and the two approximation methods were compared. The run times stated for both approximation algorithms include the time required to estimate the values of a_i and b_i. In all Fox-Landi calculations, a maximum of 128 multiplier values were examined; ten bisections were used in all applications of the bisection method. Furthermore, in both test cases all stock levels for all relevant multiplier values were stored in main memory. Thus, although the reported computation times, which include compile times, are roughly equal for all the algorithms, they are biased in favor of the Fox-Landi method because this type of storage would be impossible for larger problems. In addition, the range of multiplier values considered in the test of the Fox-Landi and bisection methods was selected after estimating the optimal multiplier value using the First Approximation Method. Thus the test results are biased in favor of them, since the range of multiplier values was much smaller than would normally be the case.

The data displayed in Tables 1 and 2 indicate how well each approach approximates a given target budget for the two test data sets. Without a doubt the bisection method produced solutions that best matched the target budgets, followed in order by the Second Approximation Method, the Fox-Landi method, and the First Approximation Method. As mentioned before, the results are biased in favor of both the Fox-Landi and bisection methods due to the initialization of the range of multiplier values. From a practical viewpoint, all approaches worked acceptably well in meeting the target budgets. Furthermore, the stock levels

Table 1

75-ITEM, 3-BASE TEST CASE

Target Budget	Total Cost ($ millions)			
	Bisection	Fox-Landi	First Approximation	Second Approximation
3.68	3.67	3.68	3.63	3.63
3.97	3.99	3.92	3.82	4.03
4.27	4.27	4.27	4.30	4.18
4.57	4.57	4.57	4.62	4.61
4.87	4.87	4.85	4.87	4.78
5.16	5.16	5.18	5.09	5.17
5.46	5.46	5.42	5.38	5.49
5.76	5.76	5.76	5.75	5.79
6.05	6.06	6.05	6.06	6.08
6.35	6.34	6.38	6.28	6.33
6.65	6.65	6.63	6.63	6.73
6.94	6.89	6.80	6.87	6.92
7.24	7.24	7.19	7.27	7.24
7.54	7.54	7.57	7.68	7.51
7.83	7.84	7.77	7.80	7.83
8.13	8.14	8.24	8.20	8.05
8.43	8.42	8.50	8.42	8.42
8.73	8.73	8.50	8.74	8.77
9.02	9.02	9.04	9.11	9.00
Execution time (sec)	92.57	19.57	11.59	4.57

generated by the various approaches were virtually the same for similar budgets. Consequently, total system expected backorders, for all practical purposes, are indistinguishable; that is, the backorder versus investment curves virtually coincide among these various approaches. Exact comparison of computed stock levels and expected backorders cannot be made among the competing methods since the allocation of the available budget in each case depends on the way each algorithm estimates the Lagrangian multiplier.

The area in which the methods clearly differ is in computation time. The approximation methods require substantially less time than either the Fox-Landi method or the more time-consuming bisection method.

Table 2

125-ITEM, 5-BASE TEST CASE

Target Budget	Total Cost ($ millions) Fox-Landi	First Approximation	Second Approximation
26.4	26.7	24.8	26.6
27.6	27.6	26.2	27.9
28.7	28.7	27.6	28.9
29.8	30.0	29.5	29.8
31.0	31.2	30.7	30.8
32.1	32.1	32.0	32.2
33.2	33.3	33.1	33.1
34.4	34.4	34.3	34.2
35.4	35.5	35.9	35.7
36.6	36.8	37.0	36.7
37.8	38.0	38.1	37.7
38.9	38.6	39.3	39.2
40.0	39.9	40.6	40.0
41.2	41.1	42.1	41.3
42.3	42.5	43.9	42.4
43.4	43.3	44.7	43.7
44.6	44.5	45.6	44.2
45.7	46.3	46.1	45.9
46.8	47.2	47.3	46.7
Execution time (sec)	36.98	16.28	4.74

NOTE: All programs are run on an IBM 370/168.

Other experimentation has shown that the percentage difference in computation times tends to be even more substantial as the number of items considered increases.

Thus the approximation methods produce answers that are as good as those produced by the Fox-Landi method and the bisection method, only much more quickly than those methods. The bisection method does, however, match target budgets slightly better than the approximation methods. However, the approximation algorithms are virtually foolproof, which is perhaps their greatest advantage. The user does not have to specify the range of multiplier values or the number of bisections in advance. This eliminates one problem associated with implementing either the Fox-Landi or bisection algorithms. In view of these observations, the approximation procedures developed here appear to be superior for use on real problems.

VI. ESTIMATION OF THE OPTIMAL DEPOT STOCK LEVEL

We have described Sherbrooke's algorithm and several Lagrangian type methods for solving problem P1, and have demonstrated that it is possible to significantly reduce the computational requirement of the Fox-Landi method by solving an approximation problem to obtain a good estimate of an appropriate value for the Lagrangian multiplier. In this section we describe a different way to reduce the computational requirements of all the algorithms that have been discussed. As can be seen by reexamining Sherbrooke's approach (see problem P6) and the Fox-Landi algorithm (see problems P8 and P9), the amount of computation required to solve problem P1 using these methods is directly proportional to the number of depot stock levels explicitly examined. Consequently, if this number can be reduced, then the total time required to compute an optimal solution can also be reduced.

The method that we describe in this section to estimate the optimal depot stock level will be of particular value when the expected number of units in the depot resupply system for an item is 20 or more. The approximation algorithm can reduce computation time for the algorithms described in Sec. IV by as much as 90 percent for these high demand items.

We have indicated how the optimal base stock level, call it s^*_{ij}, can be calculated given the depot stock level s_{i0} and the value of θ. In particular, we have shown that s^*_{ij} is optimal if it is the smallest non-negative integer for which

$$\sum_{x>s_{ij}} p(x|\lambda_{ij}T_{ij}(s_{i0})) \leq \theta c_i \ .$$

We now develop a different but equivalent way of characterizing s^*_{ij}. To simplify notation, let us suppress the item index i. We will also assume that $p(x|\lambda_j T_j(s_0))$ has a Poisson distribution.

Define the convex backorder function for base j as

$$B_j(s_j;\ s_0) \stackrel{\Delta}{=} \sum_{x>s} (x - s_j)p(x|\lambda_j T_j(s_0)) ,$$

for $s_j \geq 0$ and integer, and the piecewise linear completion of B_j, call it $\hat{B}_j$, as follows:

$$\hat{B}_j(t;\ s_0) \stackrel{\Delta}{=} \begin{cases} B_j(t;\ s_0) \quad \text{if } t \text{ is a non-negative integer.} \\ [B_j(s_j;\ s_0) - B_j(s_j - 1;\ s_0)](t - (s_j - 1)) \\ + B(s_j - 1;\ s_0),\ s_j - 1 < t < s_j ; \\ \text{where } s_j \text{ is a non-negative integer,} \\ \text{and } B(-1;\ s_0) \stackrel{\Delta}{=} \infty. \end{cases}$$

Let

$$\Delta\hat{B}_j(s_j;\ s_0) \stackrel{\Delta}{=} \hat{B}_j(s_j;\ s_0) - \hat{B}_j(s_j - 1;\ s_0)$$

when s_j is a non-negative integer, and

$$D(s_j;\ s_0) \stackrel{\Delta}{=} \{v\colon \Delta\hat{B}_j(s_j;\ s_0) < v \leq \Delta\hat{B}_j(s_j + 1;\ s_0)\} .$$

Observe that $D_j(s_j;\ s_0) \cup \{\Delta\hat{B}_j(s_j;\ s_0)\}$ is the set of subgradients of $\hat{B}_j$ at s_j. Then an alternative way of verifying that s_j^* is an optimal base stock level is to show that $-\theta c \in D(s_j^*;\ s_0)$.

Next let

$$F(s_1, s_2, \ldots, s_n;\ s_0) \stackrel{\Delta}{=} \sum_{j=1}^{m} (B_j(s_j;\ s_0) + \theta c s_j) .$$

By dropping both the integrality and non-negativity restrictions on s_0, we obtain the following relaxation of problem P8:

$$\min_{s_0} \left\{ \theta c s_0 + \min_{s_j=0,1,\ldots} \{F(s_1, \ldots, s_n;\ s_0)\colon\ s_0 \text{ fixed}\} \right\} . \quad \text{(P10)}$$

If s_0 is the optimal solution to problem P10, then

$$\frac{\partial F}{\partial s_0} + \theta c = 0 \ . \tag{E3}$$

But

$$\frac{\partial F}{\partial s_0} = \sum_{j=1}^{m} \frac{\partial B_j}{\partial T_j} \frac{\partial T_j}{\partial s_0} \ .$$

Furthermore, by writing $B_j(s_j; s_0)$ as

$$\sum_{K=1}^{\infty} Kp(K + s_j | \lambda T_j(s_0)) \ ,$$

we see that

$$\frac{\partial B_j}{\partial T_j} = \sum_{K=1}^{\infty} \lambda_j K e^{-\lambda_j T_j(s_0)} \frac{(\lambda_j T_j(s_0))^{K+s_j-1}}{(K + s_j - 1)!}$$

$$- \sum_{K=1}^{\infty} K\lambda_j e^{-\lambda_j T_j(s_0)} \frac{(\lambda_j T_j(s_0))^{K+s_j}}{(K + s_j)!}$$

$$= -\lambda_j \Delta\hat{B}(s_j; s_0) \ .$$

As we discussed in Sec. II, the function

$$B_0(s_0) \triangleq \sum_{x>s_0} (x - s_0)p(x|\lambda D)$$

can be closely approximated by an exponential function of the form $a_0 e^{-b_0 s_0}$, where a_0 and b_0 are positive real numbers. Then

$$T_j(s_0) = r_j A_j + (1 - r_j)B_j + \frac{1}{\lambda} a_0 e^{-b_0 s_0}$$

and

$$\frac{\partial T_j}{\partial s_0} = - \frac{(1 - r_j)}{\lambda} a_0 b_0 e^{-b_0 s_0} .$$

Upon combining these observations we see that

$$\frac{\partial F}{\partial s_0} \cong \sum_{j=1}^{m} \lambda_j \Delta\hat{B}(s_j;\ s_0) \frac{(1 - r_j)}{\lambda} a_0 b_0 e^{-b_0 s_0} .$$

Recall that $-\theta c \varepsilon D(s_j^*;\ s_0)$. Consequently $-\theta c$ approximates the marginal reduction in backorders at base j when the stock level at that base is s_j^*. After making this substitution and representing this further approximation of $\partial F/\partial s_0$ by $\widehat{\partial F/\partial s_0}$, we see that

$$\frac{\widehat{\partial F}}{\partial s_0} = - \sum_{j=1}^{m} (1 - r_j)\lambda_j \frac{1}{\lambda} \theta c a_0 b_0 e^{-b_0 s_0}$$

$$= -\theta c a_0 b_0 e^{-b_0 s_0} .$$

Substituting this approximation into Eq. (E3) we obtain the following estimate of the optimal depot stock level:

$$\hat{s}_0 = - \frac{1}{b_0} \ell n \left\{ \frac{1}{a_0 b_0} \right\} . \qquad \text{(E4)}$$

Recall the value of $\hat{s}_0$ is derived based on an exponential approximation of $B_0(s_0)$. As the average number of units in the depot repair cycle increases, that is, as λD increases, the quality of this exponential approximation improves in the region in which the optimal depot stock level should be located. Consequently, the approximation should be most accurate in these cases. But, the problems for which the search for the optimal depot stock level is most time-consuming for the algorithms described in Sec. IV correspond to the items having a large number of units in depot repair. Therefore, the proposed approximation method will be most appropriate for the items requiring the greatest amount of computational effort.

The approach we have described for estimating the optimal depot stock level has been coded and tested using a sample of 40 F-15 aircraft items. The test consisted of two sets of runs. In the first set, monthly flying was divided among 3 bases; in the second set the same monthly flying program was divided among 5 bases. The total budget distributed among the 40 items ranged from \$34 million to \$65 million in the first set of runs, and from \$34 million to \$88 million in the second set. Table 3 contains the data indicating both the optimal and estimated depot stock level for each item in both runs.

As shown in the table, there is usually no single optimal depot stock level for an item. Rather the optimal value depends on the amount of total item system stock available for distribution among the depot and bases. The estimate of optimal depot stock is quite close to the optimal value in all cases. Furthermore, the increase in expected system backorders using the estimated depot stock levels rather than the optimal levels is generally small. For most items the increase is substantially less than .1 backorders.

The results of the tests indicate that it is possible to estimate closely the optimal depot stock level using Eq. (E4). Additionally, incorporating this method for estimating the optimal depot stock into the algorithms described in Sec. IV will considerably reduce the search required to find the optimal depot stock level and will therefore markedly reduce the computational time needed to solve problem P1 using these algorithms.

Table 3

COMPARISON OF OPTIMAL AND ESTIMATED DEPOT STOCK LEVELS

Item	Optimal Depot Stock Levels		Estimated Optimal Depot Stock Levels
	Case I (3 bases)	Case II (5 bases)	
1	4-7	5-9	6
2	1,2	1-3	1
3	6	6,7	6
4	0-2	2,3	1
5	10,11	8-12	10
6	18-21	18-21,25	19
7	1,2	1,2	1
8	2	3,4	2
9	5,6	6,7	6
10	1	1,2	1
11	4,5	4-6	5
12	1	1	0
13	0-2	0,1	0
14	1-3	1-3	2
15	2-4	3,4	3
16	8,9	8,9	8
17	1,2	1,2	1
18	3,4	3-5	3
19	12-14	13-14	12
20	9-12	10-13	10
21	21-27	22-28	23
22	4,5	4-6	5
23	1	1-3	1
24	1,2	2,3	2
25	5-7	6,7	6
26	16	16	16
27	3	3,4	3
28	40-42	41-43	40
29	8-10	9,10	9
30	1	2	1
31	1,2	1,2	1
32	8,9	8,9	8
33	4,5	5,6	5
34	9-11	9,10	10
35	6,7	7,8	7
36	1-3	2	2
37	1,2	1,2	1
38	7,8	7-9	7
39	2,3	3,4	3
40	41-43	42-44	41

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